8.4.1

$x = 3t^2, \quad y = 2t^3, \quad (0 \le t \le 1)$

$(x')^2 = (6t)^2 = 36t^2$

$(y')^2 = (6t^2)^2 = 36t^4$

$(x')^2 + (y')^2 = 6^2 t^2 (1+t^2)$

$$\int_0^1 \sqrt{(x')^2 + (y')^2}\, dt$$

$$= \int_0^1 \sqrt{6^2 t^2 (1+t^2)}\, dt$$

$$= \int_0^1 6t\sqrt{1+t^2}\, dt$$

$$\begin{bmatrix} u = 1+t^2 \\ du = 2t\, dt \\ t = \dfrac{du}{2dt} \end{bmatrix}$$

$$= 6\int_{u(0)}^{u(1)} \frac{du}{2dt} \sqrt{u}\, dt$$

$$= 3\int_1^2 u^{1/2}\, du = 3\left[\frac{1}{1+\frac{1}{2}} u^{1+\frac{1}{2}}\right]_1^2$$

$$= 3 \cdot \frac{1}{\frac{3}{2}} \left(2^{3/2} - 1^{3/2}\right)$$

$$= 2 \cdot 2^{3/2} - 2 \cdot 1 = \sqrt{2^4 \cdot 2^1} - 2$$

$$= 4\sqrt{2} - 2$$

8.4.3

$x = a \cdot \cos^3 t, \quad y = a \cdot \sin^3 t, \quad (0 \le t \le 2\pi)$

$$(x')^2 = \left(a \cdot 3 \cdot \cos^2 t \cdot (-\sin t)\right)^2$$
$$= 9a^2 \cdot \sin^2 t \cdot \cos^4 t$$

$$(y')^2 = \left(a \cdot 3 \cdot \sin^2 t \cdot \cos t\right)^2$$
$$= 9a^2 \cdot \sin^4 t \cdot \cos^2 t$$

$$(x')^2 + (y')^2 = 9a^2 \cdot \sin^2 t \cdot \cos^2 t \cdot \underbrace{(\sin^2 t + \cos^2 t)}_{1}$$
$$= (3a \cdot \sin t \cdot \cos t)^2$$

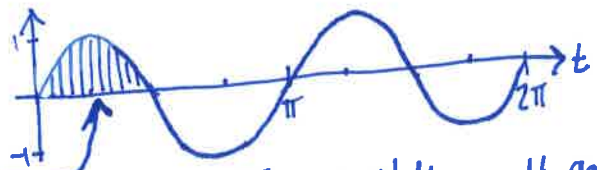
$$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt$$

$$= \int_0^{2\pi} 3a\, |\sin t \cdot \cos t|\, dt$$

$$= 3a \int_0^{2\pi} |\sin t \cdot \cos t|\, dt = 3a \int_0^{2\pi} \left|\tfrac{1}{2} \cdot \sin(2t)\right| dt$$

$$= \frac{3}{2} a \cdot \int_0^{2\pi} |\sin(2t)|\, dt$$

[$\sin(2t)$ har en periode på π:



Integralet $\int_0^{2\pi} |\sin(2t)|\, dt$ er 4 ganger dette første arealet.]

$$= \frac{3}{2} a \cdot 4 \cdot \int_0^{\pi/2} \sin(2t)\, dt = 6a \cdot \left[-\frac{1}{2} \cdot \cos(2t)\right]_0^{\pi/2}$$

$$= -3a(\cos \pi - \cos 0) = -3a(-1-1) = 6a$$

8.4.4

Se video 023

8.4.5

$x = t^2 \sin t\ ,\ y = t^2 \cos t\ ,\ (0 \le t \le 2\pi)$

$(x')^2 = (2t\sin t + t^2\cos t)^2 = 4t^2\sin^2 t + 4t^3 \sin t \cos t + t^4 \cos^2 t$

$(y')^2 = (2t\cos t - t^2 \sin t)^2 = 4t^2\cos^2 t - 4t^3 \sin t\cos t + t^4 \sin^2 t$

$$(x')^2 + (y')^2 = 4t^2 \underbrace{(\sin^2 t + \cos^2 t)}_{1} + t^4 \underbrace{(\sin^2 t + \cos^2 t)}_{1} = t^4 + 4t^2 = t^2(t^2+4)$$

$$\int_0^{2\pi} \sqrt{(x')^2+(y')^2}\,dt = \int_0^{2\pi} \sqrt{t^2(t^2+4)}\,dt = \int_0^{2\pi} t\sqrt{t^2+4}\,dt$$

$$\left[\begin{array}{l} u = t^2+4 \\ du = 2t\,dt \\ t = \dfrac{du}{2dt} \end{array}\right]$$

$$= \int_{u(0)}^{u(2\pi)} \frac{du}{2dt}\sqrt{u}\,dt = \frac{1}{2}\int_4^{4\pi^2+4} u^{1/2}\,du = \frac{1}{2}\left[\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1}\right]_4^{4\pi^2+4}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{3}{2}}\left((4\pi^2+4)^{3/2} - (4)^{3/2}\right) = \frac{1}{3}\left(\left(2^2(\pi^2+1)\right)^{3/2} - (2^2)^{3/2}\right)$$

$$= \frac{1}{3}\left((2^2)^{3/2}\cdot(\pi^2+1)^{3/2} - (2^2)^{3/2}\right) = \frac{1}{3}\left(8\cdot(\pi^2+1)^{3/2} - 8\right)$$

$$= \frac{8}{3}\left((\pi^2+1)^{3/2} - 1\right)$$

8.4.6

$x = \cos t + t\sin t, \quad y = \sin t - t\cos t, \quad (0 \le t \le 2\pi)$

$(x')^2 = (-\sin t + \sin t + t\cos t)^2 = t^2\cos^2 t$

$(y')^2 = (\cos t - \cos t + t\sin t)^2 = t^2\sin^2 t$

$(x')^2 + (y')^2 = t^2\underbrace{(\sin^2 t + \cos^2 t)}_{1} = t^2$

$\int_0^{2\pi} \sqrt{(x')^2 + (y')^2}\, dt = \int_0^{2\pi} t\, dt = \left[\frac{1}{2}t^2\right]_0^{2\pi}$

$= \frac{1}{2}\left((2\pi)^2 - 0^2\right) = 2\pi^2$

8.4.7

$x = t + \sin t, \quad y = \cos t, \quad (0 \le t \le \pi)$

$(x')^2 = (1 + \cos t)^2 = 1 + 2\cos t + \cos^2 t$

$(y')^2 = (-\sin t)^2 = \sin^2 t$

$(x')^2 + (y')^2 = 1 + 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1} = 2 + 2\cos t$

[Siden $\cos^2 t = \frac{1+\cos(2t)}{2}$, er $1 + \cos(2t) = 2\cos^2 t$
og $2 + 2\cos(t) = 4\cos^2(\frac{t}{2})$]

$\int_0^{\pi} \sqrt{(x')^2 + (y')^2}\, dt = \int_0 \sqrt{4\cos^2(\frac{t}{2})}\, dt = 2\int_0^{\pi} \left|\cos(\frac{t}{2})\right| dt$ [men $\cos(\frac{t}{2}) \geq 0$ når $(0 \le t \le \pi)$, så]

$= 2\int_0^{\pi} \cos(\frac{1}{2}t)\, dt = 2 \cdot \left[\frac{1}{1/2} \cdot \sin(\frac{1}{2}t)\right]_0^{\pi} = 4\left(\sin\frac{\pi}{2} - \sin 0\right) = 4$

8.4.8

$x = \sin^2 t,\ y = 2\cos t,\ (0 \le t \le \frac{\pi}{2})$

$(x')^2 = (2\sin t \cdot \cos t)^2 = 4\sin^2 t \cdot \cos^2 t$

$(y')^2 = (-2\sin t)^2 = 4\sin^2 t$

$(x')^2 + (y')^2 = 4\sin^2 t\,(1+\cos^2 t)$

$$\int_0^{\pi/2} \sqrt{(x')^2+(y')^2}\,dt = \int_0^{\pi/2} \sqrt{4\sin^2 t\,(1+\cos^2 t)}\,dt$$

$$= \int_0^{\pi/2} 2|\sin t| \sqrt{(1+\cos^2 t)}\,dt$$

$[\sin t \ge 0$ når $0 \le t \le \frac{\pi}{2}]$

$$= 2\int_0^{\pi/2} \sin t \sqrt{1+\cos^2 t}\,dt$$

$$\left[\begin{array}{l} u = \cos t \\ du = -\sin t\,dt \\ \sin t = -\frac{du}{dt} \end{array}\right]$$

$$= 2\int_{u(0)}^{u(\pi/2)} -\frac{du}{dt}\sqrt{1+u^2}\,dt$$

$$= -2\int_1^0 \sqrt{1+u^2}\,du = 2\int_0^1 \sqrt{1+u^2}\,du$$

Fra tabell:

$$= 2\left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln|u+\sqrt{u^2+1}|\right]_0^1$$

$$= 2\left(\left(\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln(1+\sqrt{2})\right) - \left(0 + \underbrace{\frac{1}{2}\ln 1}_{0}\right)\right)$$

$$= \sqrt{2} + \ln(1+\sqrt{2})$$

8.4.11

$x = e^t \cos t,\ y = e^t \sin t,\ (0 \le t \le \frac{\pi}{2})$

$(x')^2 = (e^t\cos t - e^t \sin t)^2 = e^{2t}(\cos t - \sin t)^2$

$(y')^2 = (e^t \sin t + e^t \cos t)^2 = e^{2t}(\cos t + \sin t)^2$

$$(x')^2 + (y')^2 = e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t + \cos^2 t + 2\sin t \cos t + \sin^2 t)$$

$$= 2e^{2t}$$

$$2\pi \int_0^{\pi/2} |e^t \cdot \sin t| \cdot \sqrt{2e^{2t}}\,dt$$

$[e^t \cdot \sin t \ge 0$ når $0 \le t \le \frac{\pi}{2}]$:

$$= 2\pi \cdot \sqrt{2} \cdot \int_0^{\pi/2} e^t \cdot \sin t \cdot e^t\,dt$$

$$= 2^{3/2}\pi \cdot \int_0^{\pi/2} e^{2t} \sin t\,dt$$

Delvis integrasjon

$$\int e^{2t} \sin t\,dt$$

$$= e^{2t}\cdot(-\cos t) - 2e^{2t}\cdot(-\sin t) + \int 4e^{2t}\cdot(-\sin t)\,dt$$

$$\int e^{2t}\sin t = \frac{1}{5}e^{2t}(2\sin t - \cos t) + C$$

$$= 2^{3/2}\pi \cdot \frac{1}{5} \cdot \left[e^{2t}(2\sin t - \cos t)\right]_0^{\pi/2}$$

$$= 2^{3/2} \cdot \frac{\pi}{5} \cdot \left(e^{\pi}(2) \underbrace{- e^0(-1)}_{+1}\right)$$

$$= \frac{2^{3/2}\pi}{5}(2e^{\pi} + 1)$$

8.4.13 Om y-aksen:

$x = 3t^2$, $y = 2t^3$, $(0 \leq t \leq 1)$

$(x')^2 = (6t)^2 = 36t^2$, $(y')^2 = (6t^2)^2 = 36t^4$

$|3t^2| = 3t^2$ når $0 \leq t \leq 1$;

$$S = 2\pi \int_0^1 3t^2 \sqrt{36t^2 + 36t^4}\, dt$$

$$= 6\pi \int_0^1 t^2 \sqrt{6^2 t^2 (1+t^2)}\, dt$$

$$= 6\pi \int_0^1 t^2 \cdot 6t \sqrt{1+t^2}\, dt$$

$$= 36\pi \int_0^1 t^3 \sqrt{1+t^2}\, dt$$

$$\left[\begin{array}{l} u = 1+t^2 \Rightarrow t^2 = u-1 \\ \frac{du}{dt} = 2t \\ dt = \frac{du}{2t} \end{array} \right]$$

$$= 36\pi \int_0^1 t^2 \cdot t \cdot \sqrt{1+t^2} \cdot dt$$

$$= 36\pi \int_{u(0)}^{u(1)} (u-1) \cdot \cancel{t} \cdot \sqrt{u} \cdot \frac{du}{2\cancel{t}}$$

$$= 18\pi \int_1^2 \left(u^{3/2} - u^{1/2}\right) du$$

$$= 18\pi \left[\frac{1}{3/2+1} u^{3/2+1} - \frac{1}{1/2+1} u^{1/2+1} \right]_1^2$$

$$= 18\pi \left[\frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right]_1^2$$

$$= 18\pi \left(\left(\frac{2}{5} \cdot 2^{5/2} - \frac{2}{3} \cdot 2^{3/2} \right) - \left(\frac{2}{5} \cdot 1^{5/2} - \frac{2}{3} \cdot 1^{3/2} \right) \right)$$

$$S = 18\pi \left(\frac{2}{5} \cdot \left(2^{\frac{4}{2}} \cdot 2^{\frac{1}{2}}\right) - \frac{2}{3} \cdot \left(2^{\frac{2}{2}} \cdot 2^{\frac{1}{2}}\right) - \frac{2}{5} + \frac{2}{3} \right)$$

$$= 18\pi \left(\frac{8}{5} \cdot 2^{\frac{1}{2}} - \frac{4}{3} \cdot 2^{\frac{1}{2}} - \frac{2}{5} + \frac{2}{3} \right)$$

$$= 18\pi \cdot 2 \left(\sqrt{2} \left(\frac{4}{5} - \frac{2}{3} \right) - \frac{1}{5} + \frac{1}{3} \right)$$

$$= 36\pi \left(\sqrt{2} \left(\frac{12-10}{15} \right) - \frac{3}{15} + \frac{5}{15} \right)$$

$$= 36\pi \left(\frac{2}{15} \sqrt{2} + \frac{2}{15} \right)$$

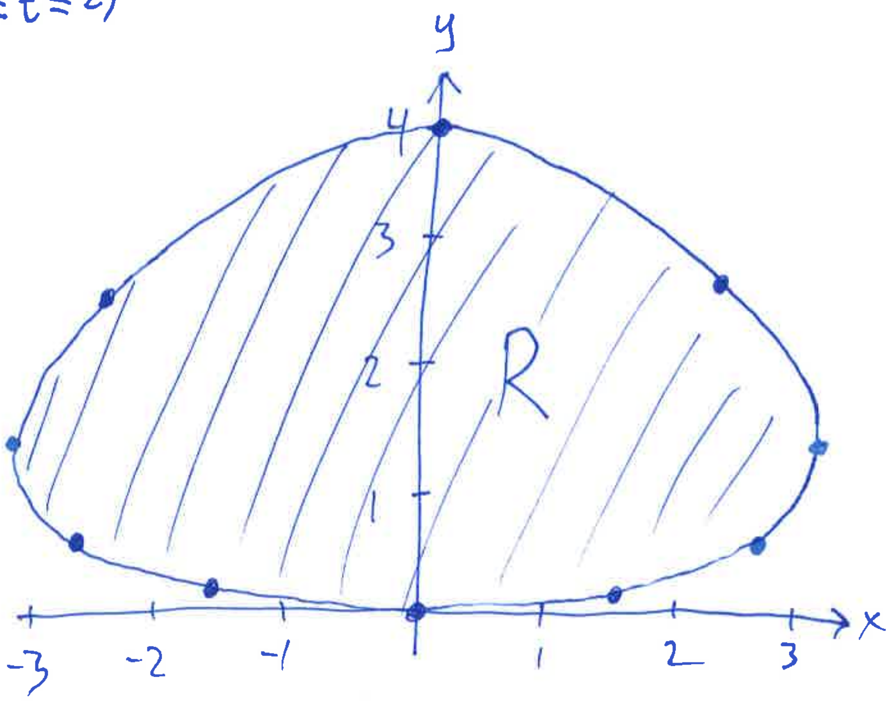
$$= \frac{72}{15} \pi \left(\sqrt{2} + 1 \right)$$

$$= \frac{24}{5} \pi \left(\sqrt{2} + 1 \right)$$

8.4.15

$x = t^3 - 4t$, $y = t^2$, $(-2 \leq t \leq 2)$

t	x	y
-2	0	4
-1.6	2.3	2.56
-1.2	3.1	1.44
-0.8	2.69	0.64
-0.4	1.54	0.16
0	0	0
0.4	-1.54	0.16
0.8	-2.69	0.64
1.2	-3.1	1.44
1.6	-2.3	2.56
2	0	4



$$R = \left| \int_{-2}^{2} g(t) \cdot f'(t)\, dt \right| \qquad \left[\begin{array}{l} f(t) = t^3 - 4t \\ f'(t) = 3t^2 - 4 \\ g(t) = t^2 \end{array} \right]$$

$$= \left| \int_{-2}^{2} t^2 (3t^2 - 4)\, dt \right|$$

$$= \left| \int_{-2}^{2} (3t^4 - 4t^2)\, dt \right|$$

$$= \left| \left[\frac{3}{5} t^5 - \frac{4}{3} t^3 \right]_{-2}^{2} \right| = \left| \left(\frac{3}{5} \cdot 32 - \frac{4}{3} \cdot 8 \right) - \left(\frac{3}{5} \cdot (-32) - \frac{4}{3} \cdot (-8) \right) \right|$$

$$= \left| 2 \cdot \frac{3}{5} \cdot 32 - 2 \cdot \frac{4}{3} \cdot 8 \right| = \left| \frac{192}{5} - \frac{64}{3} \right| = \left| \frac{256}{15} \right| = \frac{256}{15}$$